Recap an *automorphic form* $\varphi : \Gamma \backslash G \to \mathbb{C}$ $\left(\Gamma\backslash G = \begin{array}{l} SL_n\mathbb{Z} \backslash SL_n\mathbb{R} \\ GL_n\mathbb{Z} \backslash GL_n\mathbb{R} \end{array} \right)$

is a continuous function such that $\varphi(g) \ll \|g\|^m$ $\exists m$

- φ is of moderate growth
- φ is right K-finite
- φ is $\mathfrak{Z}(\mathfrak{g})$-finite

These hypotheses imply φ: analytic, $\varphi * f = \varphi$ for some $f \in C_c^\infty(G)$ (satisfying $(\cdots)$)

Defn A function $\varphi : G \to \mathbb{C}$ has *uniform moderate growth* if $\exists m \in \mathbb{R}$ s.t. $\forall D \in \mathcal{U}(\mathfrak{g})$, left-inv. diff. ops

$$D\varphi(g) \ll \|g\|^m. \quad \left(|D\varphi(g)| \leq C(m, D) \|g\|^m \right)$$

Prop Any automorphic form φ has uniform moderate growth.

Proof Choose $f \in C_c^\infty(G)$ s.t. $\varphi * f = \varphi$.

Note that $\forall D \in \mathcal{U}(\mathfrak{g})$, $D\varphi = D(\varphi * f) = \varphi * \underbrace{Df}_{\in C_c^\infty(G)}$ (general property of D, $*$)

(*Easy* if φ: moderate growth, then so is $\varphi * f$ $\forall f \in C_c(G)$ with the same exponent m)

$$\Rightarrow D\varphi(g) = (\varphi * Df)(g) \ll \|g\|^m, \quad \text{as required.} \qquad \square$$

§4. Approximation by constant terms

Simplest example: $\displaystyle\sum_{n=0}^{\infty} a_n e^{2\pi i n z} \sim a_0$ as $y \to \infty$

More generally, if φ is an automorphic form on $\Gamma\backslash G$, how does φ behave "near ∞"? (outside any compact subset)

Reduction from GL_n to SL_n

Recall that $\{$finite functions on $\mathbb{R}$ (or $\mathbb{R}_+^\times$)$\}$ = $\{$exponential polynomials$\}$

$$\left\{\begin{pmatrix} z & & \\ & \ddots & \\ & & z \end{pmatrix}\right\} \qquad \mathbb{R}_+^\times \ni y \mapsto y^\alpha (\log y)^\beta \qquad \alpha \in \mathbb{C}, \ \beta \in \mathbb{Z}_{\geq 0}$$

OR finite linear combinations

$$\| \qquad Z := \text{Center of } GL_n(\mathbb{R}) \cong \mathbb{R}^\times$$

$$\boxed{\begin{array}{c} Z \times SL_n(\mathbb{R}) \longrightarrow GL_n(\mathbb{R}) \\ \text{image of finite index} \end{array}}$$

Informally, an automorphic form on $GL_n\mathbb{Z} \backslash GL_n\mathbb{R}$ behaves in central directions like an exponential polynomial.

Lemma Let $\varphi : G \to \mathbb{C}$ be smooth, $\mathfrak{Z}(\mathfrak{g})$-finite, where $G = GL_n(\mathbb{R})$.

Then $\exists$ exponential polynomials Q_i on Z, $P_i \in \mathfrak{Z}(\mathfrak{g})$ $(i = 1..N)$

such that $\forall z \in Z, g \in G$,

$$\varphi(zg) = \sum_i Q_i(z) \cdot P_i f(g)$$

For this reason we focus hence forth on $G = SL_n(\mathbb{R})$.

Let $\mathfrak{S}$: Siegel domain for G, $\quad \Gamma \cdot \mathfrak{S} = G$.

$\mathfrak{S} = \omega A_t K$, $\quad \omega \subseteq N$: compact, $\quad t > 0$,

$$A_t = \{ a : a_i / a_{i+1} \geq t \}.$$

<u>Q</u> How can a sequence $x_\ell \in \mathfrak{S}$ $(\ell = 1, 2, \dots)$ tend off to ∞?

$x_\ell = u^{(\ell)} a^{(\ell)} k^{(\ell)}$ $\qquad \left. \begin{array}{l} u^{(\ell)} \in \omega \\ k^{(\ell)} \in K \end{array} \right\}$ compact

After passing to a subsequence, for each $i \in \{1, \dots, n-1\}$, we have either:

(1) $a_i^{(\ell)} / a_{i+1}^{(\ell)} \longrightarrow \infty$ as $\ell \to \infty$.

(2) $\sup\limits_\ell a_i^{(\ell)} / a_{i+1}^{(\ell)} < \infty$.

Let $I := \{ i : \text{possibility (2) occurs} \}$.
We also define an equivalence relation $\sim$ on $\{1, \dots, n\}$
by $i \sim j \iff \sup a_i^{(\ell)} / a_j^{(\ell)} < \infty$, $\inf a_i^{(\ell)} / a_j^{(\ell)} > 0$.

$\rightsquigarrow$ partition P of $\{1, \dots, n\}$

$\rightsquigarrow U := \{ u \in N : u_{ij} \neq 0 \Rightarrow i \not\sim j \underset{\substack{\Uparrow \\ i<j}}{\iff} a_i^{(\ell)} / a_j^{(\ell)} \xrightarrow[\ell \to \infty]{} \infty \}$

$N = \begin{pmatrix} 1 & u_{12} & u_{13} \\ & 1 & u_{23} \\ & & 1 \end{pmatrix}$ $(n=3)$

$\{1,2,3\} \supseteq I = \{1,3\}$

<u>Example</u> $G = SL_4$ $\qquad x_\ell = \begin{pmatrix} \ell & & & \\ & \ell & & \\ & & \ell^{-1} & \\ & & & \ell^{-1} \end{pmatrix}$

$P = \{ \{1,2\}, \{3,4\} \}$

$a_1^{(\ell)} / a_2^{(\ell)} = \ell / \ell = 1 \ll 1$

$a_2^{(\ell)} / a_3^{(\ell)} = \ell / \ell^{-1} = \ell^2 \to \infty$

$a_3^{(\ell)} / a_4^{(\ell)} = \ell^{-1} / \ell^{-1} = 1 \ll 1$

$U = \begin{pmatrix} 1 & 0 & * & * \\ & 1 & * & * \\ & & 1 & 0 \\ & & & 1 \end{pmatrix}$

$x_\ell = \begin{pmatrix} \ell^2 & & & \\ & \ell & & \\ & & \ell^{-1} & \\ & & & \ell^{-2} \end{pmatrix}$ $\qquad I = \emptyset$

$P = \{ \{1\}, \{2\}, \{3\}, \{4\} \}$

$U = N = \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix}$

<u>Informally</u> if φ: autom. form, then $\varphi = \varphi * f$ $\exists f \in C_c^\infty(G)$,
which ~~says~~ roughly that $\varphi(xg) \approx \varphi(x)$
for all $x \in \Gamma \backslash G$ and all <u>small</u> $g \in G$.

$\uparrow$ near 1

($\approx$: true after integrating against $f(g)\,dg$.)

<u>Lemma</u> (informal) Suppose $x_\ell \in G$. $\rightsquigarrow$ $I \subseteq \{1, \dots, n-1\}$

after passing to a subsequence

P

$U \lhd N$

Then $\varphi(u x_\ell) \approx \varphi(x_\ell)$ $\forall u \in U$.

<u>Proof</u> $x_\ell = u^{(\ell)} a^{(\ell)} k^{(\ell)}$ $\quad k^{(\ell)}, u^{(\ell)} \in$ (fixed compact).

$$u\, u^{(\ell)} = u^{(\ell)} u', \qquad u' \in U, \qquad \text{b/c } U \lhd N$$

Since φ is left-inv. under $\Gamma_U = \Gamma \cap U$, we may assume that each $|u_{ij}| \leq 1/2$ $\Rightarrow$ $u \in$ (fixed compact).

Then $u' \in$ (fixed compact).

Since A normalizes U,

$$u' a^{(\ell)} = a^{(\ell)} u'', \qquad u'' \in U.$$

$U = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ & 1 & \mathbb{R} \\ & & 1 \end{pmatrix}$

$\Gamma_U = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ & 1 & \mathbb{Z} \\ & & 1 \end{pmatrix}$

$U = \begin{pmatrix} 1 & \mathbb{R} \\ & 1 \end{pmatrix}$ $\quad \Gamma_U = \begin{pmatrix} 1 & \mathbb{Z} \\ & 1 \end{pmatrix}$

In fact,

$$u''_{ij} = \underbrace{\frac{a_j^{(\ell)}}{a_i^{(\ell)}}}_{\downarrow} \underbrace{u'_{ij}}_{\in \text{(fixed compact)}}.$$

0 if $u'_{ij} \neq 0$ (because then $i < j$, $i \not\sim j$, so $a_i^{(\ell)} / a_j^{(\ell)} \to \infty$)

$\Rightarrow$ $u''_{ij} \to 0$ as $\ell \to \infty$

$\Rightarrow$ u'': "small"

$$u x_\ell = u^{(\ell)} a^{(\ell)} \underbrace{u'' k^{(\ell)}}_{= k^{(\ell)} u'''}, \qquad u''' : \text{small } (\text{b/c } k^{(\ell)} \in K : \text{compact})$$

$\Rightarrow$ $\varphi(u x_\ell) = \varphi(x_\ell u''') \approx \varphi(x_\ell)$. $\quad$ "$\square$"

<u>Cor</u> $\varphi(x_\ell) \approx \int_{\Gamma_U \backslash U} \varphi(u x_\ell)\, du$ $\leftarrow$ probability Haar.

Defn Let $\mathcal{P}$ (or "$\sim$") be a partition of $\{1, \dots, n\}$. The corresponding <u>standard parabolic subgroup</u> $P \leq G$ is given by

$$P = \{ g \in G \cdot \ g_{ij} = 0 \ \text{ if } \ i > j, \ i \not\sim j \}$$

Ex $n = 4$:

$$\mathcal{P} = \{ \{1,2\}, \{3,4\} \} \qquad P = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix} \supset \overset{M}{\overset{\cup}{\begin{pmatrix} * & * & & \\ * & * & & \\ & & * & * \\ & & * & * \end{pmatrix}}}$$

$$\mathcal{P} = \{ \{1\}, \{2\}, \{3\}, \{4\} \} \qquad P = \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \supset \begin{pmatrix} * & & & \\ & * & & \\ & & * & \\ & & & * \end{pmatrix}$$

$$P = MU, \qquad M = \{ g \in G : \ g_{ij} \neq 0 \Rightarrow i \sim j \}$$
$$U = \{ u \in N : \ \underset{(i > j)}{u_{ij} \neq 0} \Rightarrow i \not\sim j \}$$

Defn Given $\varphi : \Gamma \backslash G \to \mathbb{C}$ (measurable, locally L^1), the <u>constant term of φ along P</u> is the function

$$\varphi_P(g) := \int_{u \in \Gamma_U \backslash U} \varphi(ug) \, du.$$

$\leftarrow$ prob. Haar

Prop Let φ: aut. form (more generally, $\varphi : \Gamma_N \backslash G \to \mathbb{C}$ of uniform moderate growth)

This easily implies $\cdot$ $\exists \ \lambda$: dominant weight s.t.

$$D\varphi(g) \ll a(g)^\lambda \quad \forall \ g \in \mathfrak{S}, \ D \in \mathcal{U}(\mathfrak{g}).$$

Then $$\varphi(g) - \varphi_P(g) \ll \sum_\beta a(g)^{\lambda - N\beta} \quad \forall \text{ fixed } N \geq 0, \quad (g \in \mathfrak{S})$$

where β runs over $\{$ weights of the form $a^\beta = a_i / a_j$, $\underset{i \not\sim j}{i < j}$, $\}$

$$= \{ \text{eigenvalues for } A \underset{\mathrm{Ad}}{\circlearrowright} \mathrm{Lie}(U) \}.$$

Defn A function $\varphi : \Gamma \backslash G \to \mathbb{C}$ is called *cuspidal* if $\varphi_P = 0$ $\forall$ standard parabolic subgroups $P \subsetneq G$.

A *cusp form* is a cuspidal automorphic form

Cor of Prop Any cusp form on $SL_n\mathbb{Z} \backslash SL_n\mathbb{R}$ is of (uniform) rapid decay, hence bounded, $\in L^2$, $(\cdots)$.

Pf It suffices to check on some $\mathfrak{S}$ on a sequence x_ℓ. We may pass to subsequences. We obtain I, $\mathfrak{P}$, $P = MU$ as before$^{(*)}$ and apply Prop. Then each $a(g)^{-N\beta}$ is very small. $\square$